

MAS231 Geometry II  
Coursework #5 Solutions

1. If  $\gamma(t) = \left( \int_0^t \cos(\theta(u)) du, \int_0^t \sin(\theta(u)) du, 0 \right)$

then  $\dot{\gamma}(t) = (\cos \theta(t), \sin \theta(t), 0)$

$$\Rightarrow \|\dot{\gamma}(t)\|^2 = \cos^2 \theta(t) + \sin^2 \theta(t) + 0 = 1$$

$\Rightarrow \gamma$  is unit speed.

$$\gamma(0) = (0, 0, 0)$$

and  $\dot{\gamma}(0) = (\cos \theta(0), \sin \theta(0), 0)$  where  $\theta(0) = 0$

$$\Rightarrow \dot{\gamma}(0) = (1, 0, 0)$$

$$\ddot{\gamma}(t) = (-\dot{\theta}(t) \sin(\theta(t)), \dot{\theta}(t) \cos(\theta(t)), 0)$$

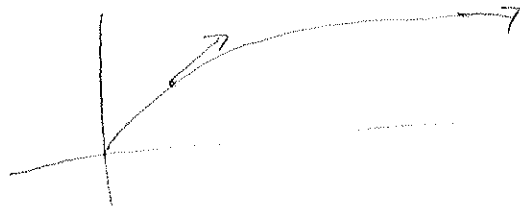
~~the~~  $\ddot{\gamma}(t) = \sqrt{K(t)} \dot{\gamma}(t) = (-K(t) \sin(\theta(t)), K(t) \cos(\theta(t)), 0)$

$$\Rightarrow \ddot{\gamma}(0) = (-K(0) \sin(0), K(0) \cos(0), 0) = (0, K(0), 0)$$

and ~~the~~  $\|\ddot{\gamma}\| = \sqrt{(-K(t) \sin(\theta(t)))^2 + (K(t) \cos(\theta(t)))^2 + 0}$   
 $= K(t)$

By the theorem about the uniqueness of unit speed curves having a given  $\gamma(0)$ ,  $\dot{\gamma}(0)$ ,  $\ddot{\gamma}(0)$  and curvature  $K(t)$ ,  $\gamma(t)$  is the unique unit speed curve such that  $\gamma(0) = (0, 0, 0)$ ,  $\dot{\gamma}(0) = (1, 0, 0)$ ,  $\ddot{\gamma}(0) = (0, K(0), 0)$  and ~~and~~ having curvature  $K(t)$ .

2. a)



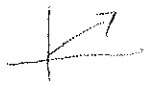
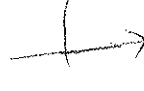
$$b) \quad \dot{\mathbf{r}}(t) = \left(1, \frac{1}{2\sqrt{t}}, 0\right) \quad \tau(t) = \frac{\dot{\mathbf{r}}}{\|\dot{\mathbf{r}}\|} = \frac{1}{\sqrt{1 + \frac{1}{4t}}} \left(1, \frac{1}{2\sqrt{t}}, 0\right)$$

$$\tau\left(\frac{1}{4}\right) = \frac{1}{\sqrt{2}} (1, 1, 0)$$

$$\lim_{t \rightarrow \infty} \tau(t) = (1, 0, 0)$$

$$c) \quad \ddot{\mathbf{r}}(t) = \left(0, -\frac{1}{4t^{3/2}}, 0\right) \quad \dot{\mathbf{r}} \times \ddot{\mathbf{r}} = -\frac{1}{4t^{3/2}} \hat{\mathbf{k}}$$

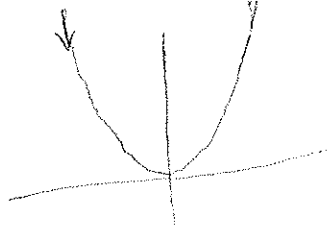
$$\frac{\dot{\mathbf{r}} \times \ddot{\mathbf{r}}}{\|\dot{\mathbf{r}}\|^3} = \frac{-\frac{1}{4t^{3/2}} \hat{\mathbf{k}}}{\left(1 + \frac{1}{4t}\right)^{3/2}} \Rightarrow K_S = \frac{-\frac{1}{4t^{3/2}}}{\left(1 + \frac{1}{4t}\right)^{3/2}}$$

Angle change is  this to  this

$$\text{or } -\frac{\pi}{4}$$

$$\begin{aligned} \text{Thus } \int_{\frac{1}{4}}^{\infty} K_S \|\dot{\mathbf{r}}\| dt &= \int_{\frac{1}{4}}^{\infty} \frac{-\frac{1}{4t^{3/2}}}{\left(1 + \frac{1}{4t}\right)^{3/2}} \left(1 + \frac{1}{4t}\right)^{1/2} dt \\ &= \int_{\frac{1}{4}}^{\infty} -\frac{1}{4t^{3/2} \left(1 + \frac{1}{4t}\right)} dt = \int_{\frac{1}{4}}^{\infty} -\frac{1}{\sqrt{t}(4t+1)} dt = -\frac{\pi}{4} \\ &\Rightarrow \int_{\frac{1}{4}}^{\infty} \frac{1}{\sqrt{t}(4t+1)} dt = \frac{\pi}{4} \end{aligned}$$

5. a)



$$b) \quad \vec{r}(t) = (1, 4t^3, 0) \quad \vec{v}(t) = \frac{1}{\sqrt{1+16t^6}} (1, 12t^2, 0)$$

$$\lim_{t \rightarrow \infty} \vec{v}(t) = (0, 1, 0)$$

$$\lim_{t \rightarrow -\infty} \vec{v}(t) = (0, -1, 0)$$

c) Change in angle is  $\downarrow$  this to  $\uparrow$  this or  $\pi$

$$\vec{r}''(t) = (0, 12t^2, 0)$$

$$\vec{r}' \times \vec{r}'' = 12t^2 \hat{k}$$

$$\|\vec{r}'\|^3 = (1+16t^6)^{3/2}$$

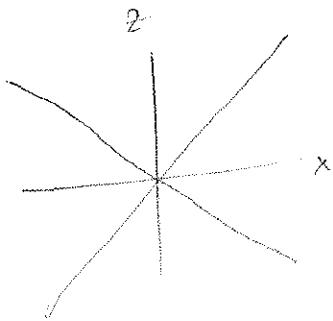
$$K_S = \frac{12t^2}{(1+16t^6)^{3/2}}$$

Hence  $\int_{-\infty}^{\infty} \frac{12t^2}{(1+16t^6)^{3/2}} (1+16t^6)^{1/2} dt = \pi$

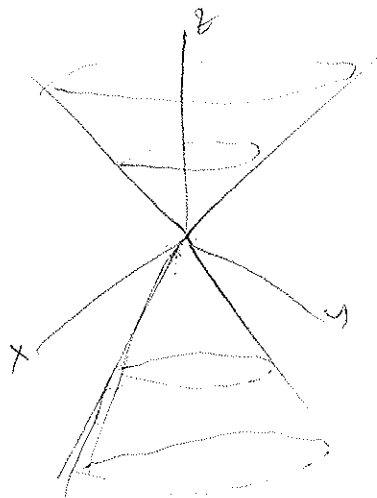
$$\Rightarrow \int_{-\infty}^{\infty} \frac{12t^2}{1+16t^6} dt = \pi$$

4-a) If  $z=c$ , then  $x^2+y^2=c^2$ , so the intersection is a circle of radius  $|c|$  centred at  $(0,0,c)$ .

b) If  $y=0$ , then  $x^2-z^2=0$   
 $x^2=z^2$   
 so  $z=\pm x$



c)  $S$  is obtained by rotating its intersection with the  $x-z$  plane about the  $z$  axis.



$S$  is a cone above the  $x-y$  plane along with its reflection below the  $x-y$  plane.